Module 2

- **N.B.**
  - If for any rule, (n) is the index no.(or sub-intervals), then the rule uses (n+1) nodes or pts.
  - **∇= Nabla.**
- **Numerical Differentiat^{n} ↓**
- **Forward diff. formula for derivative of order O(h)**

$$f'(x) \approx \frac{\Delta f(x)}{h} = \frac{f(x+h) - f(x)}{h}$$

Here, T(h)= O(h)

- **Central diff. formula for derivative of order O(h^2)**

$$f'(x) \approx \frac{\delta f(x)}{2h} = \frac{f(x+h) - f(x-h)}{2h}$$

  - **Reas^{n} of O(h^2) due to Truncat^{n} error**

$$T(h) = \frac{-h^2 f'''(\xi)}{3!}$$

Here, T(h)= O(h^2)

- **Backward diff. formula for derivative**

$$f'(x) \approx \frac{\nabla f(x)}{h} = \frac{f(x) - f(x-h)}{h}$$

- **Forward diff. formula for (k)th order derivative**

$$f^k(x) = \frac{\Delta^k f(x)}{h^k}$$

- **Backward diff. formula for (k)th order derivative**

$$f^k(x) = \frac{\nabla^k f(x)}{h^k}$$

- Central diff. formula for (k)th order derivative

$$f^k(x) = \frac{\delta^k f(x)}{h^k}$$

  - For *f''(x):*

$$f''(x) \approx \frac{\delta^2 f(x)}{h^2} = \frac{f(x-h) - 2f(x) + f(x+h)}{h^2}$$

- **Numerical Integrat^n ↓**

  Also called **numerical quadrature** or **mech. quadrature** or **approx. evaluat^n of integrals**.

- Riemann sum (n-pt. integrat^n rule)

$$\delta = R_n(f) = \sum_{n=1}^{k} a_n f(x_n)$$

- Rectangular rule
  - Left

$$R'(f) = (b-a)f(a)$$

  - Right

$$R''(f) = (b-a)f(b)$$

- Mid-pt. rule

$$M(f) = (b-a)f\left(\frac{a+b}{2}\right)$$

- **Trapezoidal rule**

$$T(f) = \frac{1}{2}(b-a)(f(a)+f(b))$$

$$T(f) = \frac{R'(f)+R''(f)}{2}$$

- **Newton-Cotes rule**

$$I(P_n) = \sum_{n=0}^{k} \omega_n f(x_n) = R_{n+1}(f)$$

  - To calculate ω_i

$$\omega_i = \int_a^b l_i(x)dx$$

$$\omega_i = \frac{h \cdot (-1)^{n-i}}{i!(n-i)!}\left(\int_0^n \alpha(\alpha-1)\dots(\alpha-i+1)(\alpha-i-1)\dots(\alpha-n)d\alpha\right)$$

  - **Error**

$$E(f) = \frac{1}{(n+1)!}\left(\int_a^b (x-x_0)\dots(x-x_n)f^{n+1}(\eta)dx\right)$$

- **Trapezoidal rule (n=1)**

$$I(P) = T(f) = \frac{h}{2}(f(a)+f(b))$$

$$h = \frac{b-a}{1}$$

• Here, no. of nodes is 2.

- Error

$$E(f) = -\frac{h^3}{12} f''(\eta)$$

- N.B.
  - If n is even:

$$E(f) = \frac{h^{n+3} f^{n+2}(\eta_1)}{(n+2)!} \left(\int_0^n \mu^2 (\mu - 1) \dots (\mu - n) d\mu \right)$$

Here, h=(b-a)/n

  - If n is odd:

$$E(f) = \frac{h^{n+2} f^{n+1}(\eta_2)}{(n+1)!} \left(\int_0^n \mu (\mu - 1) \dots (\mu - n) d\mu \right)$$

Here, h=(b-a)/n

- Simps^{n}'s (1/3) rule (n=2)

$$I(P) = \frac{h}{3} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right]$$

$$h = \frac{b-a}{2}$$

• Also called **Parabolic Rule**.
• Here, no. of nodes is 3.

  - Error

$$E(f) = -\frac{(h)^5}{90} f^{iv}(\eta)$$

- Simps^{n}'s (3/8) rule (n=3)

$$I(P) = \frac{3h}{8} [f(x_0) + 3(f(x_1) + f(x_2)) + f(x_3)]$$

$$h = \frac{b-a}{3}$$

• Here, no. of nodes is 4.

- Error

$$E(f) = -\frac{3h^5}{80} f^{iv}(\eta)$$

- Relat^n between S(1/3)(f), M(f) & T(f)

$$S_{\frac{1}{3}}(f) = \frac{2M(f) + T(f)}{3}$$

- Compound Left Rectangular rule

$$I_{CLR}(f) = h[f(x_0) + f(x_1) + \ldots + f(x_{n-1})]$$

$$h = \frac{b-a}{n}$$

- Error

$$E_{CLR}(f) = \frac{nh^2}{2} f'(\mu)$$

- Compound Rt. Rectangular rule

$$I_{CRR}(f) = h[f(x_1) + f(x_2) + \ldots + f(x_n)]$$

$$h = \frac{b-a}{n}$$

- **Compound Mid-pt. rule**

$$I_{CMR}(f) = 2h[f(x_1) + f(x_3) + \ldots + f(x_{2n-1})]$$

$$h = \frac{b-a}{2n}$$

• Here, odd no. of nodes are needed for getting even no. of sub-intervals.
• Here, 2n is the no. of sub-intervals.

  - **Error**

$$E_{CMR}(f) = \frac{nh^3}{3} f''(\mu)$$

- **Compound Trapezoidal rule**

$$I_{CTR}(f) = \frac{h}{2}[f(x_0) + 2(f(x_1) + f(x_2) + \ldots + f(x_{n-1})) + f(x_n)]$$

$$h = \frac{b-a}{n}$$

  - **Error**

$$E_{CTR}(f) = -\frac{nh^3}{12} f''(\mu)$$

- **Compound Simps^{n}'s (1/3) rule**

$$I_{CS\frac{1}{3}}(f) = \frac{h}{3}\left[f(x_0) + 2\left(\sum_{n=2} f(x_{2n-2})\right) + 4\left(\sum_{n=1} f(x_{2n-1})\right) + f(x_{2n})\right]$$

$$h = \frac{b-a}{2n}$$

• Here, odd no. of nodes are needed for getting even no. of sub-intervals.
• Here, 2n is the no. of sub-intervals.

- Error

$$E_{cs\frac{1}{3}}(f) = -\frac{nh^5}{90} f^{iv}(\mu)$$

- Gauss-Legendre (Gauss Quadrature) rule

$$t = \frac{2x - (b + a)}{b - a}$$

$$x = \left(\frac{b - a}{2}\right) t + \left(\frac{b + a}{2}\right)$$

• If the range isn't in [-1, 1], then apply the above 2 formulas.
• An (n+1) gauss rule has degree of precisⁿ as (2n+1).

- 2-pt. form

$$R_2(f) = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

Degree of precisⁿ is 3.

- 3-pt. form

$$R_3(f) = \frac{1}{9}\left[\, 8f(0) + 5f\left(\frac{\sqrt{3}}{\sqrt{5}}\right) + 5f\left(-\frac{\sqrt{3}}{\sqrt{5}}\right) \,\right]$$

Degree of precisⁿ is 5.

- **Numerical Differential equn.↓**

Deals with 1st order differential equn.

- Picard's method

$$y_{n+1} = y_0 + \int_{x_0}^{x} f(x, y_n) dx$$

- **Euler's method**

$$y_{n+1} = y_n + hf(x_n, y_n)$$

- **Modified Euler's Method**

$$y_{n+1} = Y_n + hf(x_n, Y_n)$$
$$Y_{n+1} = Y_n + \frac{h}{2}[f(x_n, Y_n) + f(x_{n+1}, y_{n+1})]$$

- **2nd order Runge-Kutta method**
  - **Improved tangent method**

$$y_{n+1} = y_n + hf\left(x_n + \frac{h}{2}, y_n + \frac{h}{2}f(x_n, y_n)\right)$$

    Also called **2 pt. RK method** or **mid-pt. method**.

  - **Heun's method**

$$y_{n+1} = y_n + \frac{h}{2}[f(x_n, y_n) + f(x_{n+1}, y_n + hf(x_n, y_n))]$$

    Also called **Modified Euler's method**.

  - **Optimal method**

$$y_{n+1} = y_n + \frac{h}{4}\left[f(x_n, y_n) + 3f\left(x_n + \frac{2h}{3}, y_n + \frac{2h}{3}f(x_n, y_n)\right)\right]$$

- **3rd order Runge-Kutta method**

Here, n=0,1,2,3,...

  - **Heun's formula**

$$y_{n+1} = y_n + \frac{1}{4}(K_1 + 3K_3)$$

$$\text{where, } K_1 = hf(x_n, y_n)$$

$$K_2 = hf\left(x_n + \frac{h}{3}, y_n + \frac{K_1}{3}\right)$$

$$K_3 = hf\left(x_n + \frac{2h}{3}, y_n + \frac{2K_2}{3}\right)$$

- Classical formula

$$y_{n+1} = y_n + \frac{1}{6}(K_1 + 4K_2 + K_3)$$

$$\text{where, } K_1 = hf(x_n, y_n)$$

$$K_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{K_1}{2}\right)$$

$$K_3 = hf(x_n + h, y_n - K_1 + 2K_2)$$

- 4th order Runge-Kutta method

Here, n=0,1,2,3,...

- Kutta's formula

$$y_{n+1} = y_n + \frac{1}{8}(K_1 + 3K_2 + 3K_3 + K_4)$$

where, $K_1 = hf(x_n, y_n)$

$$K_2 = hf\left(x_n + \frac{h}{3}, y_n + \frac{K_1}{3}\right)$$

$$K_3 = hf\left(x_n + \frac{2h}{3}, y_n - \frac{K_1}{3} + K_2\right)$$

$$K_4 = hf(x_n + h, y_n + K_1 - K_2 + K_3)$$

- Classical formula

$$y_{n+1} = y_n + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

where, $K_1 = hf(x_n, y_n)$

$$K_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{K_1}{2}\right)$$

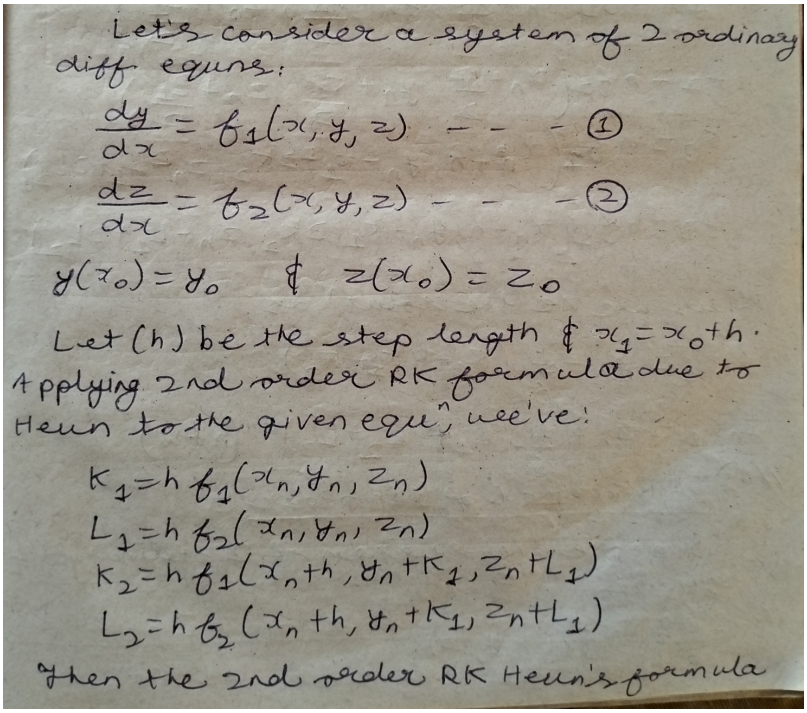
$$K_3 = hf\left(x_n + \frac{h}{2}, y_n + \frac{K_2}{2}\right)$$

$$K_4 = hf(x_n + h, y_n + K_3)$$

- **System of numerical differential equns.↓**

  Deals with the system of 1st order differential equns.

- Using Heun's formula of 2nd order RK method


Let's consider a system of 2 ordinary diff equns:

$$\frac{dy}{dx} = f_1(x, y, z) \quad - - - \quad (1)$$

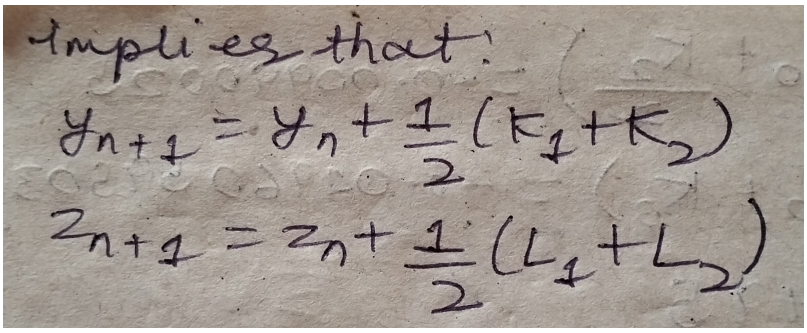
$$\frac{dz}{dx} = f_2(x, y, z) \quad - - - \quad (2)$$

$y(x_0) = y_0$ & $z(x_0) = z_0$

Let (h) be the step length & $x_1 = x_0 + h$.
Applying 2nd order RK formula due to Heun to the given equn, we've:

$$K_1 = h f_1(x_n, y_n, z_n)$$
$$L_1 = h f_2(x_n, y_n, z_n)$$
$$K_2 = h f_1(x_n + h, y_n + K_1, z_n + L_1)$$
$$L_2 = h f_2(x_n + h, y_n + K_1, z_n + L_1)$$

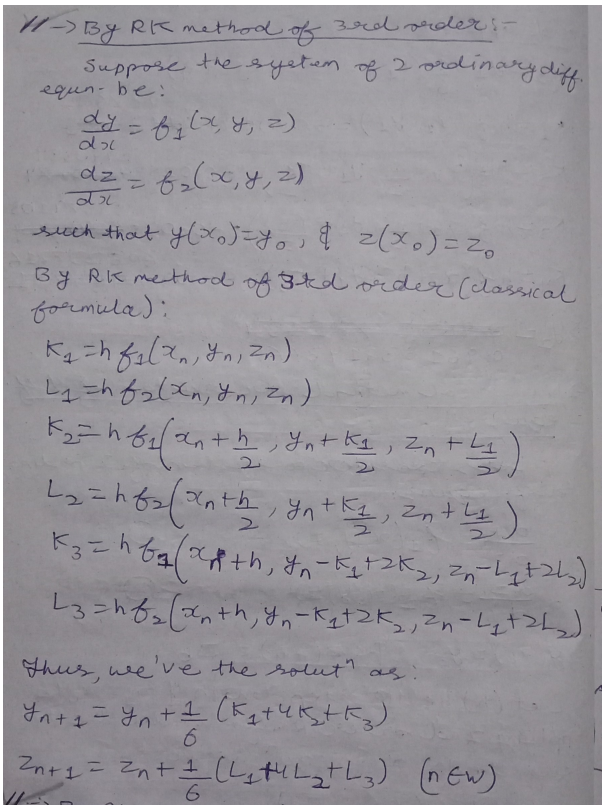
Then the 2nd order RK Heun's formula


Implies that:

$$y_{n+1} = y_n + \frac{1}{2}(K_1 + K_2)$$

$$z_{n+1} = z_n + \frac{1}{2}(L_1 + L_2)$$

- Using Classical formula of 3rd order RK method


// -> By RK method of 3rd order:-

Suppose the system of 2 ordinary diff equn be:

$$\frac{dy}{dx} = f_1(x, y, z)$$

$$\frac{dz}{dx} = f_2(x, y, z)$$

such that $y(x_0) = y_0$, & $z(x_0) = z_0$

By RK method of 3rd order (classical formula):

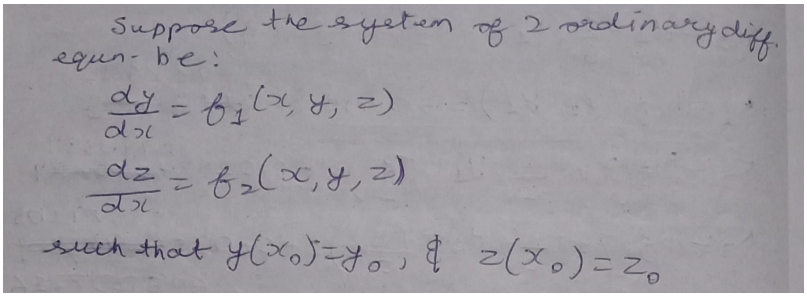
$$K_1 = h f_1(x_n, y_n, z_n)$$
$$L_1 = h f_2(x_n, y_n, z_n)$$
$$K_2 = h f_1\left(x_n + \frac{h}{2}, y_n + \frac{K_1}{2}, z_n + \frac{L_1}{2}\right)$$
$$L_2 = h f_2\left(x_n + \frac{h}{2}, y_n + \frac{K_1}{2}, z_n + \frac{L_1}{2}\right)$$
$$K_3 = h f_1(x_n + h, y_n - K_1 + 2K_2, z_n - L_1 + 2L_2)$$
$$L_3 = h f_2(x_n + h, y_n - K_1 + 2K_2, z_n - L_1 + 2L_2)$$

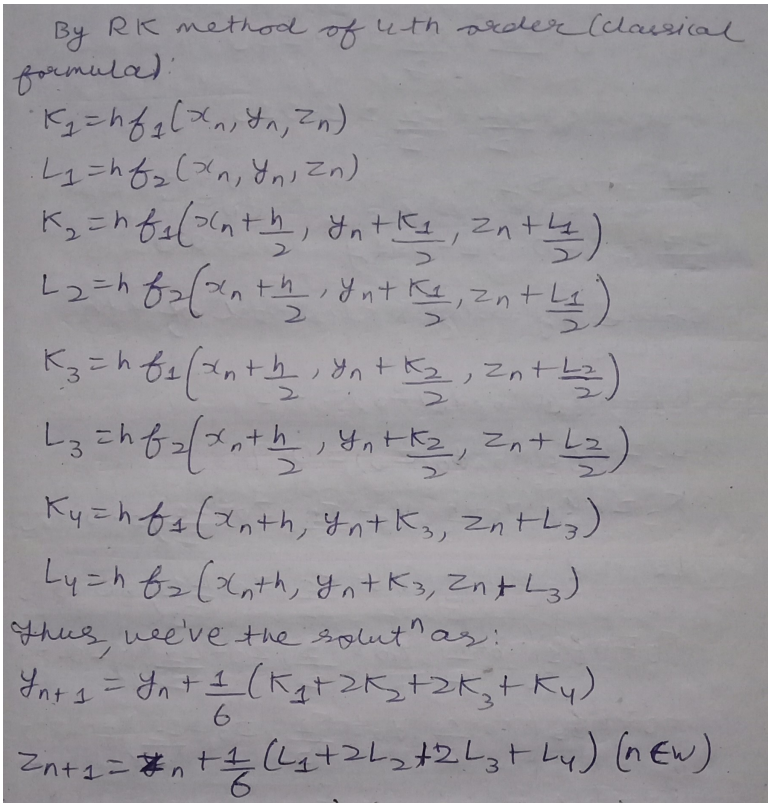
Thus, we've the solutn as:

$$y_{n+1} = y_n + \frac{1}{6}(K_1 + 4K_2 + K_3)$$

$$z_{n+1} = z_n + \frac{1}{6}(L_1 + 4L_2 + L_3) \quad (n \in W)$$

- Using Classical formula of 4th order RK method




By RK method of 4-th order (classical formula):
$K_1 = h f_1(x_n, y_n, z_n)$
$L_1 = h f_2(x_n, y_n, z_n)$
$K_2 = h f_1(x_n + \frac{h}{2}, y_n + \frac{K_1}{2}, z_n + \frac{L_1}{2})$
$L_2 = h f_2(x_n + \frac{h}{2}, y_n + \frac{K_1}{2}, z_n + \frac{L_1}{2})$
$K_3 = h f_1(x_n + \frac{h}{2}, y_n + \frac{K_2}{2}, z_n + \frac{L_2}{2})$
$L_3 = h f_2(x_n + \frac{h}{2}, y_n + \frac{K_2}{2}, z_n + \frac{L_2}{2})$
$K_4 = h f_1(x_n + h, y_n + K_3, z_n + L_3)$
$L_4 = h f_2(x_n + h, y_n + K_3, z_n + L_3)$
Thus, we've the solutn as:
$y_{n+1} = y_n + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$
$z_{n+1} = z_n + \frac{1}{6}(L_1 + 2L_2 + 2L_3 + L_4)$ $(n \in W)$

- **Multi-step method ↓**

  Deals with 1st order differential equn.

- **Milnes method**

  - **Milnes Predictor formula**

    $$y_{n+1} = y(x_{n+1}) = y_{n-3} + \frac{4h}{3}[2y'_{n-2} - y'_{n-1} + 2y'_n]$$

    - **Ex: (4=0123)**

      $$y_4 = y_0 + \frac{4h}{3}[2y'_1 - y'_2 + 2y'_3]$$

  - **Milnes Corrector formula**

    $$y^*_{n+1} = y(x_{n+1}) = y_{n-1} + \frac{h}{3}[y'_{n-1} + 4y'_n + y'_{n+1}]$$

    - **Ex: (4=2234)**

$$y^*_4 = y_2 + \frac{h}{3}[y'_2 + 4y'_3 + y'_4]$$

- Adam's method
  - Adam's-Basforth predictor formula

$$y_{n+1} = y_n + \frac{h}{24}[55y'_n - 59y'_{n-1} + 37y'_{n-2} - 9y'_{n-3}]$$

It's an **explicit** formula, as it doesn't include y(n+1) term on the RHS.

- **Ex: (4=33210)**

$$y_4 = y_3 + \frac{h}{24}[55y'_3 - 59y'_2 + 37y'_1 - 9y'_0]$$

  - Adam's-Moulton corrector formula

$$y^*_{n+1} = y_n + \frac{h}{24}[9y'_{n+1} + 19y'_n - 5y'_{n-1} + y'_{n-2}]$$

It's an **implicit** formula, as it includes y(n+1) term on the RHS.

- **Ex: (4=34321)**

$$y^*_4 = y_3 + \frac{h}{24}[9y'_4 + 19y'_3 - 5y'_2 + y'_1]$$